Exam Waves and Optics – 3 February 2012

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Questions and answers

Question 1

Show how right-circularly polarised light can be obtained from the superposition of two linearly polarised harmonic waves with perpendicular polarisation directions. First give the expressions for the two individual waves and their superposition. Then show that this superposition represents right-circularly polarised light.

(Section 8.1.2 from Hecht)

take 2 linearly polarised waves with perpendicular polarisation directions, such that the wave:

- run in the same direction (take e.g. the positive z-axis)
- have a phase difference of $-\frac{\pi}{2}$
- have the same amplitude

in formulae:

$\bar{E}_x(z,t) = \hat{\imath}\, E_0 \cos(kz-\omega t)$ polarised along the x-axis

$\bar{E}_y(z,t) = \hat{\jmath}\, E_0 \cos(kz-\omega t-\frac{\pi}{2})$ " " " y- "

$= \hat{\jmath}\, E_0 \sin(kz-\omega t)$

superposition: $\bar{E}(z,t) = E_0[\hat{\imath}\cos(kz-\omega t) + \hat{\jmath}\sin(kz-\omega t)]$

show that this represents right-circular polarised light:

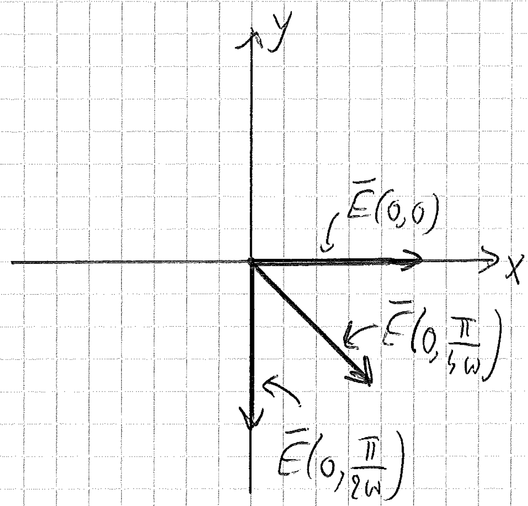
consider the point $z=0$, looking into the wave (in the negative z-direction)

$\bar{E}(0,t) = E_0[\hat{i}\cos\omega t - \hat{j}\sin\omega t]$

$\omega t = 0: \bar{E}(0,0) = E_0\hat{i}$

$\omega t = \frac{\pi}{4}: \bar{E}(0,\frac{\pi}{4\omega}) = E_0(\hat{i}\frac{1}{\sqrt{2}} - \hat{j}\frac{1}{\sqrt{2}})$

$\omega t = \frac{\pi}{2}: \bar{E}(0,\frac{\pi}{2\omega}) = -E_0\hat{j}$



$\bar{E}$ turns clockwise. This is, according to convention, right-circular polarized light

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Question 2

The superposition of N waves with the same frequency (ω) but with different amplitudes (E_{0i}) and initial phases (α_i), $E = \sum_{i=1}^{N} E_{0i}\, cos(\alpha_i \pm \omega t)$, can be written as:

$$E = E_0 \cos(\alpha \pm \omega t)$$

where:

$$E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2\sum_{j>i}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}\cos(\alpha_i - \alpha_j)$$

$$\tan\alpha = \frac{\sum_{i=1}^{N} E_{0i}\sin\alpha_i}{\sum_{i=1}^{N} E_{0i}\cos\alpha_i}$$

Assume that all waves have the same amplitude: $E_{0i} = E_{01}$. What is then the intensity of the superposition, expressed as a function of E_{01} and N, for:

a) non-coherent waves;

b) coherent waves in a point where all waves are in phase ?

Describe the results in the context of the principle of conservation of energy.

(page 285 - 286 in Hecht)

intensity of the superposition $\propto \langle E_0^2 \rangle_T$ (time average of E_0^2)

non-coherent waves

the phase of the waves changes very quickly and there is no relationship between the phases of the individual waves

so: $\langle \cos(\alpha_i - \alpha_j) \rangle_T = 0$, for all i, j

$$E_0^2 = N E_{01}^2$$

the total energie ($\propto$ intensity) = $N \times$ energy of 1 wave

energy is conserved in each point in space

coherent waves, in a point where all waves are in phase

wave in phase: $\alpha_i = \alpha_j$, for all i, j

$$\cos(\alpha_i - \alpha_j) = 0$$

$$E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2 \sum_{j>i}^{N} \sum_{i=1}^{N} E_{0i} E_{0j}$$

$$= \left(\sum_{i=1}^{N} E_{0i} \right)^2$$

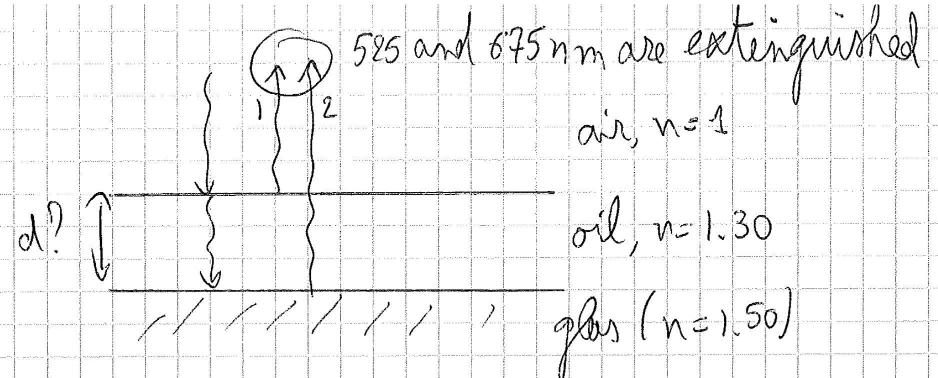
$$E_{0i} = E_{01} \Rightarrow E_0^2 = (N E_{01})^2 = N^2 E_{01}^2$$

The energy in the point considered is larger than the sum of the energies of the individual waves. From conservation of energy it then follows that, in other points (where the phases are not equal), the energy will be smaller than the sum. Globally, energy is conserved, but the spatial distribution is different than for non-coherent waves.

Question 3

White light impinges perpendicularly on a thin oil film (refractive index 1.30) on a glass substrate. The reflected light is extinguished at a wavelength of both 525 and 675 nm.

1. How thick is the oil film ?
2. Which orders of interference minima are involved ?



rays 1 and 2 both result from external reflection, so there is no relative phase difference due to reflection

light is extinguished when the difference in optical path length:

$$\Delta OPL = (m+\tfrac{1}{2})\lambda \qquad m = 0, \pm 1, \pm 2, \ldots$$

(wave have phase difference of π and interfere destructively)

difference in OPL between reflected light 1 and 2: $2nd$

so $2nd = (m+\frac{1}{2})\lambda \rightarrow d = (m+\frac{1}{2})\frac{\lambda}{2n}$

this condition needs to be fulfilled for $\lambda_1 = 525$ and $\lambda_2 = 675$ nm

$$(m_1+\tfrac{1}{2})\frac{\lambda_1}{2n} = (m_2+\tfrac{1}{2})\frac{\lambda_2}{2n}$$

solution 1: from the above: $m_1 = 1.286\, m_2 + 0.143$

integer numbers that fulfil this:

$m_2 = 3, \; m_1 = 4$

so: 3rd order minimum for 675 nm $\Rightarrow d = \frac{3.5 \times 675}{2 \times 1.30} =$ 909 nm

4th " " " 525 nm

Solution 2: calculate d for both wavelengths and different orders (see below): for $m_1 = 4$ and $m_2 = 3$, the same d, 909 nm, is obtained

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orde	525 nm	675 nm
0	100.96	129.81
1	302.88	389.42
2	504.81	649.04
3	706.73	**908.65**
4	**908.65**	1168.27
5	1110.58	1427.88
6	1312.50	1687.50

Question 4

A plane wave with intensity I_0 and wavelength λ impinges on a rectangular aperture in the *y,z*-coordinate plane. The intensity in a point *P* at a distance r_0 past the aperture is then given by:

$$I_p = \frac{I_0}{4}\{[\mathscr{C}(u_2) - \mathscr{C}(u_1)]^2 + [\mathscr{S}(u_2) - \mathscr{S}(u_1)]^2\}$$
$$\times \{[\mathscr{C}(v_2) - \mathscr{C}(v_1)]^2 + [\mathscr{S}(v_2) - \mathscr{S}(v_1)]^2\}$$

$\mathscr{C}$ and $\mathscr{S}$ are the Fresnel integrals. The variables *u* and *v* are related to the coordinates *y* and *z* in the aperture (relative to the perpendicular from point *P* on the plan of the aperture:

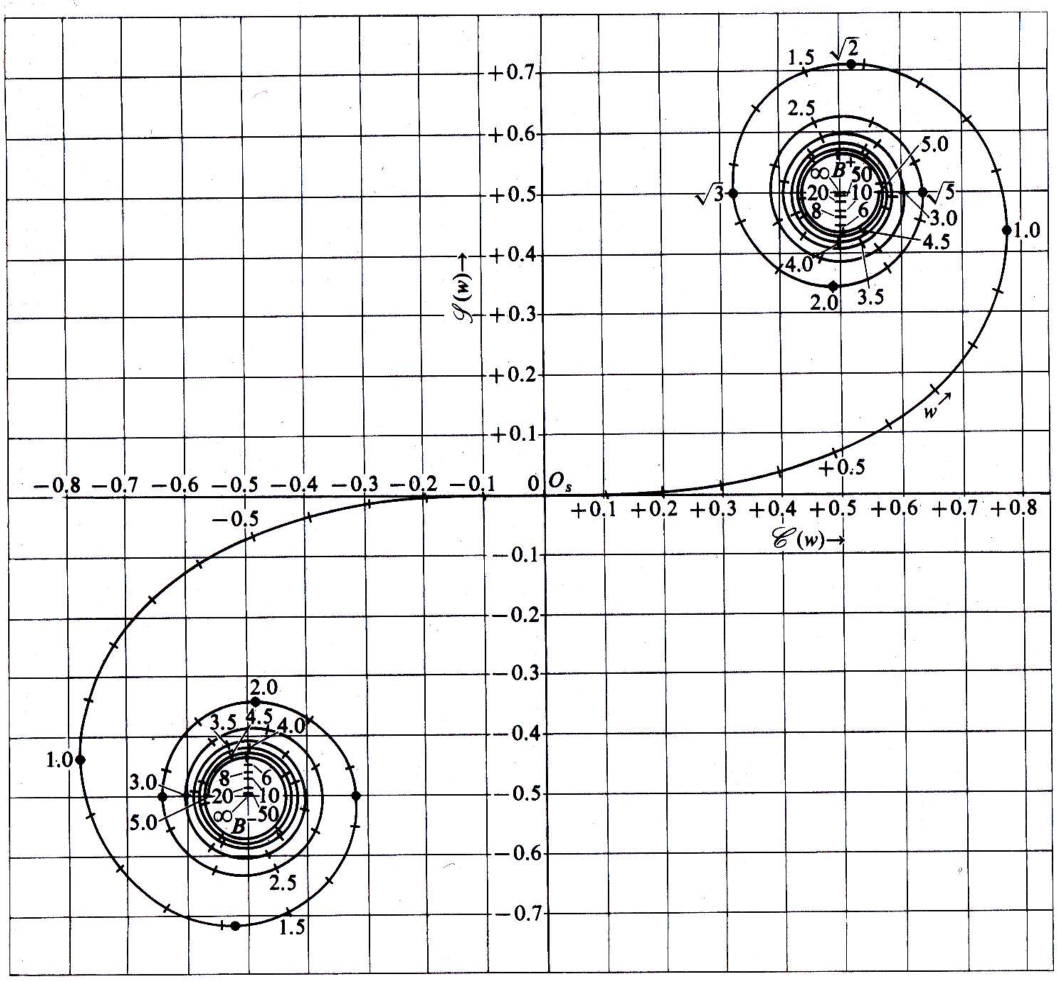
$$u \equiv y\left[\frac{2}{\lambda r_0}\right]^{1/2} \qquad v \equiv z\left[\frac{2}{\lambda r_0}\right]^{1/2}$$

u_1, u_2 and v_1, v_2 correspond to the edges of the aperture in the *y* and *z* directions.

Now consider a square aperture with sides of 2 mm that is being illuminated by a plane wave with intensity I_0 and wavelength λ = 500 nm. The point *P* is located at a distance r_0 = 4 m past the aperture, perpendicular on the middle of the aperture.

1. Show that in this situation Fresnel diffraction takes place, and not Fraunhofer diffraction.

2. Determine the light intensity in the point P. For this, use the Cornu-spiral shown below (the variable w has the meaning of either u or v, depending on the coordinaat being considered).



The Cornu spiral.

1. Fraunhoferdiffraction if $R > \frac{a^2}{\lambda}$, otherwise Fresnel diffraction
R = minimum distance source-obstacle or obstacle-screen
a = dimension obstacle (aperture in this case)
λ = wavelength of the light

$a = 2$ mm, $\lambda = 500\ 10^{-6}$ mm, $R = 4000$ mm

$$\frac{a^2}{\lambda} = \frac{4}{500\ 10^{-6}} = 8000 \text{ mm} > R \Rightarrow \text{Fresnel diffraction}$$

2. P is perpendicular to the centre =

$y_1 = -1$ mm , $y_2 = 1$ mm

$z_1 = -1$ mm , $z_2 = 1$ mm

$$u_1 = -1\left(\frac{2}{500\ 10^{-6}\ 4000}\right)^{1/2} = -1 \quad , \quad u_2 = 1$$

$$v_1 = -1 \qquad\qquad v_2 = 1$$

$\mathcal{C}(-1) \approx -0.78$ $\qquad$ $\mathcal{C}(1) \approx 0.78$

$\mathcal{S}(-1) \approx -0.43$ $\qquad$ $\mathcal{S}(1) \approx 0.43$

filling this out in the formula given:

$$I_p = \frac{I_0}{4}\left\{(2\times0.78)^2+(2\times0.43)^2\right\}\left\{(2\times0.78)^2+(2\times0.43)^2\right\}$$

$$= 2.52\ I_0$$